Entropy Production in Relativistic Hydrodynamics

H.-Th. Elze[1,2], J. Rafelski[1] and L. Turko[1,3]

[1] Department of Physics, University of Arizona, Tucson, AZ 85721

[2] Instituto de Fisica, Universidade Federal do Rio de Janeiro
C.P. 68528, 21945-970 Rio de Janeiro, RJ, Brasil

[3] Institute of Theoretical Physics, University of Wroclaw
pl. Maksa Borna 9, 50-204 Wroclaw, Poland

Abstract

The entropy production occurring in relativistic hydrodynamical systems such as the quark-gluon plasma (QGP) formed in high-energy nuclear collisions is explored. We study mechanisms which change the composition of the fluid, *i.e.* particle production and/or chemical reactions, along with chemo- and thermo-diffusion. These effects complement the conventional dissipative effects of shear viscosity, bulk viscosity, and heat conductivity.

Introduction. There is fundamental interest in the study of high energy density matter, which promises to illucidate further the properties of the strongly interacting vacuum and the high temperature phases of strongly interacting matter [1, 2]. The method of choice for the description of the rapidly evolving and possibly inhomogeneous quark-gluon plasma (QGP) has been often the hydrodynamical approach [3, 4]. It is generally believed that QGP existed shortly after the Big Bang initial phase of the universe, and that it may be re-created in relativistic nuclear collisions [5, 6].

In order to gain a better understanding of this new form of matter, we explore here the entropy production due to numerous dissipative effects arising in the evolution of QGP towards the point of hadronization. The entropy content of the QGP phase at hadronization determines the observable particle multiplicity [7]. Therefore, in order to establish via particle multiplicity the initial conditions reached in heavy-ion collisions [3], the effect of entropy production in QGP evolution has to be understood.

To begin with, we present here the main result of this paper, before embarking on its derivation. The entropy production is given by:

$$
\begin{aligned}
T\partial_\mu s^\mu &= -\sum_i \mu_i \mathcal{J}_i - T\sum_{i,j} \sigma_{ij}(\partial_\mu \frac{\mu_i}{T})\triangle^{\mu\nu}(\partial_\nu \frac{\mu_j}{T}) \\
&\quad -\frac{\kappa}{T}\mathcal{Q}_\mu \triangle^{\mu\nu}\mathcal{Q}_\nu + \frac{\eta}{2}\mathcal{W}_\alpha^\beta \triangle_\beta^\gamma \mathcal{W}_\gamma^\delta \triangle_\delta^\alpha + \zeta(\partial_\mu u^\mu)^2 \ ,
\end{aligned}
\tag{1}
$$

see Eq. (23). We will show that the four-divergence of the entropy current, $\partial_\mu s^\mu$, is always nonnegative and vanishes only, when all space-time gradients of temperature T, of chemical potentials μ_i, and of four-velocity u_μ vanish, and particle densities of species "i" assume their chemical equilibrium values. – The projector $\triangle$ is: $\triangle_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu$, with $g_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$ and $u^2 = 1$; we use units such that $\hbar = c = k_B = 1$.

The first two terms on the right-hand side of Eq. (1) are new. They are due to the sources $\mathcal{J}_i$ of particle production and chemo- and thermo-diffusion contributions involving gradients of chemical potentials and temperature , respectively. These terms arise in the continuity Eq. (6) below; σ denotes the matrix of mutual diffusion coefficients between the chemical QGP components. Furthermore, the constants κ, η, and ζ denote the transport coefficients of heat conductivity, shear viscosity, and bulk viscosity, respectively. The corresponding terms in Eq. (1) involve the heat-flow four-vector $\mathcal{Q}_\mu$, Eq. (17), and the shear tensor $\mathcal{W}_{\mu\nu}$, Eq. (18), and have been obtained earlier [4, 8].

Generally, dissipative processes are important, when space-time gradients of hydro- or

thermodynamical quantities in the system become large relative to its relaxation scales. The latter and especially the transport coefficients and source terms introduced in Eq. (1) have to be fit to experiment or calculated from an underlying microscopic transport theory [8, 9]. For a recent extensive review of such state-of-the-art microscopic calculations and new gauge theory results we refer to Ref. [10] and its numerous references.

In the QGP case, particle production converting kinetic energy of a nuclear collision into high multiplicity of new particles is a characteristic feature, together with annihilation processes, changing the composition of the plasma [11, 12, 13]. Motivated by this fact, our aim presently has been to incorporate these dissipative and entropy producing mechanisms consistently into the relativistic hydrodynamical framework.

Equations of Motion. Presently we consider the relativistic energy-momentum tensor for an imperfect fluid:

$$T^{\mu\nu} \equiv (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + \delta T^{\mu\nu} = T^{\nu\mu} \ , \tag{2}$$

where ϵ and P denote its local energy density and pressure, respectively. The additional term $\delta T^{\mu\nu}$ incorporates the dissipative effects within the fluid [4, 8]. It will be obtained in detail in the following. We point out that the corresponding transport coefficients have to be determined consistently, *i.e.*, including also all contributions from particle production or other composition changing processes, if present in the system, besides the scattering contributions [9, 10].

The evolution of the hydrodynamical system then is governed by the equation of motion,

$$\partial_\mu T^{\mu\nu} = f^\nu \ , \tag{3}$$

where f^μ describes the density of external forces, gravity for example, which act on the fluid locally. A closed system has $f^\mu \equiv 0$, which results in total four-momentum conservation. In order to obtain a complete description of the fluid, the equation of motion (2) has to be supplemented by an equation of state relating the energy density to the pressure. In the simplest case of a noninteracting ultrarelativistic system, we have $P = \epsilon/3$, while a realistic QGP equation of state is given in Ref. [14].

Often, particularly if there exist conserved charges or particle numbers, a simple functional relationship between pressure and energy density does not exist. Then, assuming local thermal and possibly chemical equilibrium, both can still be represented parametrically in

terms of a temperature $T = T(x)$ and possibly several chemical potentials $\mu_i = \mu_i(x)$. In this case, one has to complement Eqs. (2)-(3) by continuity equations,

$$\partial_\mu \rho_i^\mu \equiv \partial_\mu \rho_i u^\mu = 0 \ , \tag{4}$$

where $\rho_i \equiv \rho_i(x)$ denotes the particle number density for species "i" in the local rest frame of a fluid cell. We will not distinguish the cases of charge or particle number conservation and count each (anti-)particle species separately. – The local rest frame is always determined by $u^\mu = u_{(0)}^\mu \equiv (1, 0, 0, 0)$. Thus, by definition, we have $\rho_{i(0)}^0 = \rho_i$.

Similarly, the total energy density is defined to be equal to ϵ. Therefore, we must require that dissipative terms do not contribute to T^{00} in the local rest frame,

$$T_{(0)}^{00} = \epsilon \ \longrightarrow \ \delta T_{(0)}^{00} = 0 \ \longrightarrow \ u_\mu u_\nu \delta T^{\mu\nu} = 0 \ , \tag{5}$$

cf. Eq. (2). This constraint plays an important role in determining the general structure of $\delta T^{\mu\nu}$ consistently with the Second Law of Thermodynamics.

In order to incorporate diffusion and particle production and/or chemical reactions among the constituents, which change the composition of the fluid, we now generalize Eq. (4) to read:

$$\partial_\mu \tilde{\rho}_i^\mu \equiv \partial_\mu(\rho_i^\mu + \triangle^{\mu\nu}\partial_\nu \mathcal{R}_i) = \mathcal{J}_i \ , \tag{6}$$

with the new four-current $\tilde{\rho}_i$ having the property $u_\mu \tilde{\rho}_i^\mu = \rho_i$, since $u_\mu \triangle^{\mu\nu} = 0$. Furthermore, $\mathcal{J}_i$ denotes the local source for particle species "i", which is a functional of the densities, $\mathcal{J}_i(x) \equiv \mathcal{J}_i[\rho_j](x)$. It can be calculated in terms of microscopic reaction cross sections. We do not consider here the possibility of external sources. – In a homogeneous system such continuity equations become familiar chemical rate equations, which have previously been applied to strongly interacting relativistic systems on the hadronic and partonic level [11, 15, 16].

The explicit form of the diffusion term $\propto \partial \mathcal{R}_i$ in Eq. (6) will be determined consistently with the Second Law of Thermodynamics. – One might try to follow the nonrelativistic ansatz which recovers the usual diffusive contribution to the particle current in Eq. (6) in the local rest frame:

$$\mathcal{R}_i \equiv \sum_j \tilde{\sigma}_{ij}\rho_j \ , \quad \vec{j}_i \equiv -\nabla \mathcal{R}_i = -\sum_j \tilde{\sigma}_{ij}\nabla\rho_j \ , \tag{7}$$

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where $\tilde{\sigma}_{ij}$ represents the mutual diffusion coefficient between species "i" and "j". However, this ansatz turns out to be incompatible with the requirement that the entropy never decreases and will be suitably generalized below.

Entropy Production in an Imperfect Fluid. Here we formulate a consistent theoretical framework to study the entropy production due to dissipative effects within a relativistic fluid. We assume local thermal equilibrium and characterize the particle densities by appropriate chemical potentials.

We remark that introducing chemical potentials does *not* necessarily imply chemical equilibrium in the sense of saturating the available phase space [17]. – For example, consider a thermalized electron-positron plasma with one chemical potential regulating the charge density, as usual, and with a number of e^+e^--pairs which does not correspond to the given temperature. Such a situation can be described by introducing a phase space occupancy factor in addition to the charge chemical potential or, equivalently, by two chemical potentials regulating the number of electrons and positrons separately. – We presently admit such cases, recall Eq. (6), which typically involve charge conservation side by side with dynamically changing particle pair populations in the fluid.

In order to obtain a relationship between the thermodynamical quantities and variables discussed so far and the entropy, in particular, we employ the (local) equilibrium relations,

$$- PV = \Omega(T, V, \mu_i) = U - TS - \sum_i \mu_i N_i \ , \tag{8}$$

where U, S, and N_i denote the internal energy, entropy, and particle numbers, respectively.

Two comments are in order here. Firstly, having nuclear collisions in mind, we employ the grandcanonical description despite the fact that the fireball created in a single event is thermodynamically closed. There is no heat or particle bath. However, instead of generating the grandcanonical ensemble by our system being connected to such a bath, we consider a large number of events to form the ensemble. The events have to be identical with respect to the conserved total energy-momentum, angular momentum, and baryon number, to name the most relevant "macroscopic" observables. Nevertheless, microscopically, the initial state may differ from event to event. Then, under the assumption that the entropy is maximized, the grandcanonical ensemble is obtained [17, 18]. In the hydrodynamical approach we apply this reasoning locally to each fluid cell.

Secondly, consider finite size effects present in small volume systems. In this case, the

usual ($V \to \infty$) statistical description breaks down, if the unphysical fluctuations of exactly constrained quantities are not eliminated.[1] This issue needs to be addressed in our context in the future.

We also need the Duhem-Gibbs relation for the densities corresponding to Eq. (8),

$$Ts = \epsilon + P - \sum_i \mu_i \rho_i \ . \tag{9}$$

Then, using Eqs. (2) and (9), we obtain by contraction with u_ν of the equation of motion (3):

$$
\begin{aligned}
u_\nu f^\nu &= u_\nu \partial_\mu T^{\mu\nu} = u_\nu \partial_\mu ([Ts + \sum_i \mu_i \rho_i] u^\mu u^\nu) - u^\mu \partial_\mu P + u_\nu \partial_\mu \delta T^{\mu\nu} \\
&= u_\nu u^\nu \partial_\mu [T\tilde{s}^\mu + \sum_i \mu_i \rho_i^\mu] + [T\tilde{s}^\mu + \sum_i \mu_i \rho_i^\mu] u_\nu \partial_\mu u^\nu - u^\mu \partial_\mu P + u_\nu \partial_\mu \delta T^{\mu\nu} \\
&= \partial_\mu [T\tilde{s}^\mu + \sum_i \mu_i \rho_i^\mu] - u^\mu \partial_\mu P + u_\nu \partial_\mu \delta T^{\mu\nu} \ , \tag{10}
\end{aligned}
$$

where we introduced an entropy four-current $\tilde{s}^\mu \equiv s u^\mu$ and the currents $\rho_i^\mu \equiv \rho_i u^\mu$, as before. For the last equality, we made use of $u^2 = 1$.

For conservative external forces which can be derived from a potential (density), $f^\mu \equiv \partial^\mu \phi$, using Eq. (6) and $u^\mu \partial_\mu = d/d\tau$, we can further simplify Eq. (10) ($\dot{A} \equiv dA/d\tau$):

$$
\begin{aligned}
0 &= T \partial_\mu \tilde{s}^\mu + \sum_i \mu_i (\mathcal{J}_i - \partial_\mu \triangle^{\mu\nu} \partial_\nu \mathcal{R}_i) + s\dot{T} + \sum_i \rho_i \dot{\mu}_i - \dot{P} - \dot{\phi} + u_\nu \partial_\mu \delta T^{\mu\nu} \\
&= T \partial_\mu \tilde{s}^\mu + \sum_i \mu_i (\mathcal{J}_i - \partial_\mu \triangle^{\mu\nu} \partial_\nu \mathcal{R}_i) + u_\nu \partial_\mu \delta T^{\mu\nu} \ , \tag{11}
\end{aligned}
$$

where the terms involving $d/d\tau$ cancel identically. In order to demonstrate this cancellation, we multiply them by $V \cdot d\tau$ and employ Eq. (8),

$$
\begin{aligned}
& SdT + \sum_i N_i d\mu_i - VdP - Vd\phi \\
&= d(TS) + d(\sum_i \mu_i N_i) - d(PV) - TdS - \sum_i \mu_i dN_i + PdV - Vd\phi \\
&= dU - TdS - \sum_i \mu_i dN_i + PdV - Vd\phi = 0 \ . \tag{12}
\end{aligned}
$$

The last equality expresses the First Law of Thermodynamics, which is incorporated here.

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[1]Such constraints arise due to exact conservation laws, unbroken internal symmetries, and quantization of the motion of the constituents (see, e.g., Refs. [19, 20, 21, 22]). For example, the thermodynamical properties of small quark-gluon plasma droplets start to deviate strongly from the asymptotic ($V \to \infty$) behavior, if the dimensionless parameter $TV^{1/3}$ becomes of order one or less.

Finally, we introduce the proper entropy four-current defined by:

$$s^\mu \equiv \tilde{s}^\mu + T^{-1} u_\nu \delta T^{\mu\nu} + \sum_i \mathcal{L}_i \triangle^{\mu\nu} \partial_\nu \mathcal{R}_i \ . \tag{13}$$

This form is found by 'trial and error' and, by the constraint (5) and $u_\mu \triangle^{\mu\nu} = 0$, implies $s^0_{(0)} = s$ in the local rest frame.

We remark that the difficulty to identify a viable form of the entropy four-current, *i.e.* the admissible structure of $\delta T^{\mu\nu}$, $\mathcal{L}_i$, and $\mathcal{R}_i$, resides in the requirements that the entropy density be equal to s in the local rest frame and that it never decreases anywhere in the system. In the following we will determine the function $\mathcal{L}_i$ consistently together with $\mathcal{R}_i$ and $\delta T^{\mu\nu}$, in order to obtain the new contributions due to diffusion and, particularly, due to particle production, which have not been considered before.

We combine Eqs. (11) and (13), which yields:

$$
\begin{aligned}
T \partial_\mu s^\mu &= -\sum_i \mu_i \mathcal{J}_i - T^{-1} (\partial_\mu T) u_\nu \delta T^{\mu\nu} + (\partial_\mu u_\nu) \delta T^{\mu\nu} \\
&\quad + \sum_i (\mu_i + T\mathcal{L}_i) \partial_\mu \triangle^{\mu\nu} \partial_\nu \mathcal{R}_i + T \sum_i (\partial_\mu \mathcal{L}_i) \triangle^{\mu\nu} \partial_\nu \mathcal{R}_i \ .
\end{aligned}
\tag{14}
$$

This equation describes the entropy production in an imperfect fluid evolving under the influence of conservative external forces, which do not generate entropy (Liouville's theorem). In order to agree with the Second Law of Thermodynamics for the closed system ($f^\mu \equiv 0$), the right-hand side of Eq. (14) has to be nonnegative for all fluid configurations, such that $\partial_\mu s^\mu \geq 0$ always. This requirement is implemented in the following steps.

First, we put a constraint on the sources introduced in Eq. (6),

$$\sum_i \mu_i \mathcal{J}_i \leq 0 \ . \tag{15}$$

This constraint and its implications will be further discussed below, beginning with Eq. (24). Under fairly general assumptions about the particle producing processes or chemical reactions the inequality (15) guarantees that the system is driven towards chemical equilibrium.

Second, the structure of $\delta T^{\mu\nu}$ including terms of first order space-time gradients, which is subject to the constraint (5) and compatible with $\partial_\mu s^\mu \geq 0$, is well known [8]:

$$\delta T^{\mu\nu} = \kappa (\triangle^{\mu\gamma} u^\nu + \triangle^{\nu\gamma} u^\mu) \mathcal{Q}_\gamma + \eta \triangle^{\mu\gamma} \triangle^{\nu\delta} \mathcal{W}_{\gamma\delta} + \zeta \triangle^{\mu\nu} \partial_\gamma u^\gamma \ , \tag{16}$$

where $\kappa, \eta, \zeta \geq 0$ denote the coefficients of heat conductivity, shear viscosity, and bulk viscosity, respectively, and with the heat-flow four-vector $\mathcal{Q}_\mu$ and the shear tensor $\mathcal{W}_{\mu\nu}$

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defined by:

$$\mathcal{Q}_\mu \equiv \partial_\mu T - T u^\nu \partial_\nu u_\mu \ , \tag{17}$$

$$\mathcal{W}_{\mu\nu} \equiv \partial_\mu u_\nu + \partial_\nu u_\mu - \frac{2}{3} g_{\mu\nu} \partial_\gamma u^\gamma \ . \tag{18}$$

Third, as we shall see in Eq. (23), the contribution to the right-hand side of Eq. (14) of the diffusion terms involving $\mathcal{R}$ becomes a nonnegative quadratic form if we identify:

$$\mathcal{L}_i \equiv -\frac{\mu_i}{T} \ , \quad \mathcal{R}_i \equiv \sum_j \sigma_{ij} \frac{\mu_j}{T} \ , \tag{19}$$

where the symmetric matrix σ consists of the mutual diffusion constants and supposedly has only nonnegative eigenvalues. We remark that $\mathcal{R}_i$ as defined in (19) yields a diffusive particle current in the local rest frame,

$$\vec{j}_i \equiv -\nabla \mathcal{R}_i = -\sum_j \frac{\sigma_{ij}}{T} (\nabla \mu_j - \frac{\mu_j}{T} \nabla T) \ . \tag{20}$$

Thus, we find here a combination of chemo- and thermo-diffusion contributions, which appears to be unique up to terms involving first-order space-time gradients.

We observe that the diffusion current of Eq. (20) indeed generalizes the nonrelativistic ansatz discussed before, cf. Eqs. (7). For example, neglecting interactions, a straightforward calculation in the ultrarelativistic Boltzmann approximation gives:

$$\mu_j = T \ln[C \rho_j T^{-3}] \ , \tag{21}$$

where C collects the constants. Using this in Eq. (20), $i.e.$ $\mu_j = \mu_j(\rho_j, T)$, we obtain:

$$\vec{j}_i = -\sum_j \sigma_{ij} (\frac{\nabla \rho_j}{\rho_j} + 3\frac{\nabla T}{T}) \ . \tag{22}$$

Obviously the term $\propto \nabla \rho_j$ corresponds to the nonrelativistic one, while our covariant considerations additionally yield an unique thermo-diffusion term.

Collecting the results of this section, we can now rewrite Eq. (14) with all the sources of entropy production in an explicitly nonnegative form:

$$
\begin{aligned}
T \partial_\mu s^\mu &= -\sum_i \mu_i \mathcal{J}_i - T \sum_{i,j} \sigma_{ij}^{-1} (\partial_\mu \mathcal{R}_i) \triangle_\lambda^\mu (\partial_\nu \mathcal{R}_j) \triangle^{\nu\lambda} \\
&\quad -\frac{\kappa}{T} \mathcal{Q}_\mu \triangle_\lambda^\mu \mathcal{Q}_\nu \triangle^{\nu\lambda} + \frac{\eta}{2} \triangle_\alpha^\beta \mathcal{W}_\beta^\gamma \triangle_\gamma^\delta \triangle_\delta^\mu \mathcal{W}_\mu^\nu \triangle_\nu^\alpha + \zeta (\partial_\mu u^\mu)^2 \ ,
\end{aligned} \tag{23}
$$

which is obtained with the help of Eqs. (16)-(19) and by making use of the properties of the projector $\triangle$. All transport coefficients are treated as constants here, despite the fact that they implicitly depend on the thermodynamical variables, if calculated microscopically [10].

Together with condition (15), the Eq. (23) guarantees that a closed system evolves from local towards global equilibrium. The latter is reached, of course, when all space-time gradients of T, μ_i, and u^μ vanish and the particle densities have their asymptotic values, such that the sources $\mathcal{J}_i$ vanish.

Driving Towards Chemical Equilibrium. We now turn to the discussion of the local source terms $\mathcal{J}_i[\rho_j]$, which describe the particle production in the continuity equations (6). This will clarify the meaning of the constraint (15).

We assume a sufficiently small fluid cell with a given volume, such that all spatial gradients can be neglected in Eq. (6). This does not limit our considerations which are related to the invariant source terms $\mathcal{J}_i$. Thus, in the local rest frame, we obtain approximately:

$$\partial_t \rho_i = \mathcal{J}_i \equiv \sum_j [G_{i \leftarrow j} \rho_j - L_{j \leftarrow i} \rho_i] \;, \tag{24}$$

where the source term is expressed in terms of gain and loss rates, G and L, respectively. In general, they are complicated function(al)s of the temperature and the densities themselves (see, for example, Refs. [11, 15, 16] and Chap. X. in Ref. [18]).

The master equations (24) form a first order nonlinear flow system [23]. In the following we will show that Eq. (15) presents a sufficient condition for an attractive fixed point of the flow, which corresponds to the local chemical equilibrium.

In order to proceed, we introduce the equilibrium densities $\bar{\rho}_i(T) \equiv \rho_i(\bar{\mu}_i, T)$, where $\bar{\mu}_i$ denotes the corresponding equilibrium chemical potential, and the reduced densities $\xi_i \equiv \rho_i/\bar{\rho}_i$, $0 \leq \xi_i < \infty$. We assume that

$$\mu_i(\xi_i, T) \equiv \bar{\mu}_i + f_i(\xi_i, T) \;, \tag{25}$$

with $f_i(1, T) = 0$, is a strictly increasing function of ξ_i at a given temperature T. This is exemplified by the result obtained in the Boltzmann approximation, Eq. (21),

$$\mu_i(\xi_i, T) = \bar{\mu}_i + T\ln[\xi_i] \;. \tag{26}$$

In any case, the point $\vec{\xi}_1 \equiv \{\xi_i = 1, \text{ for all } i\}$ in the multidimensional ξ-space corresponds to the chemical equilibrium state.

Sufficiently close to equilibrium, $\vec{\xi} \approx \vec{\xi}_1$, the constraint (15) can be expanded. We suppress the temperature dependence from now on, tacitly assuming (as before) that local

thermal equilibrium is maintained by sufficiently fast elastic scattering among the constituents of the considered fluid cell. For convenience, we introduce rates weighted by the equilibrium densities, $\bar{G}_{i\leftarrow j} \equiv G_{i\leftarrow j}\bar{\rho}_j$ and $\bar{L}_{j\leftarrow i} \equiv L_{j\leftarrow i}\bar{\rho}_i$. Then, employing Eq. (25) and using principle of detailed balance,

$$\sum_j [\bar{G}_{i\leftarrow j}(\vec{\xi}_1) - \bar{L}_{j\leftarrow i}(\vec{\xi}_1)] = 0 \ , \tag{27}$$

i.e. $\mathcal{J}_i = 0$ in chemical equilibrium, we obtain:

$$0 \leq -\sum_i \mu_i \mathcal{J}_i \approx -\sum_{i,j} \bar{\mu}_i(\vec{\xi} - \vec{\xi}_1) \cdot \nabla(\bar{G}_{i\leftarrow j}\xi_j - \bar{L}_{j\leftarrow i}\xi_i)|_{\vec{\xi}_1} \ . \tag{28}$$

The first nonvanishing contribution here is a positive gradient term; in the absence of equilibrium chemical potentials one obtains a curvature term here instead. Therefore, the nonnegative contribution $-\sum_i \mu_i \mathcal{J}_i$ to the entropy production, Eq. (23), vanishes in the chemical equilibrium state.

In order to prove that the equilibrium point $\vec{\xi} = \vec{\xi}_1$ is actually an attractive fixed point, we suitably rewrite the flow system, Eqs. (24),

$$\partial_t(\xi_i - 1) = \bar{\rho}_i^{-1} \sum_j [\bar{G}_{i\leftarrow j}\xi_j - \bar{L}_{j\leftarrow i}\xi_i] \ , \tag{29}$$

and introduce a distance measure $\mathcal{M}$,

$$\mathcal{M}(\vec{\xi},\vec{\xi}_1) \equiv \frac{1}{2} \sum_i \bar{\rho}_i f_i'(1)(\vec{\xi} - \vec{\xi}_1)_i(\vec{\xi} - \vec{\xi}_1)_i \geq 0 \ , \tag{30}$$

since $f_i' \equiv df_i/d\xi_i > 0$, μ_i being a strictly increasing function. Then, we obtain:

$$\partial_t \mathcal{M}(\vec{\xi},\vec{\xi}_1) = \sum_i (\mu_i(\xi_i) - \bar{\mu}_i)\bar{\rho}_i \partial_t(\xi_i - 1) = \sum_i \mu_i(\xi_i)\mathcal{J}_i(\vec{\xi}) \leq 0 \ , \tag{31}$$

which holds sufficiently close to the equilibrium state, where $\mu_i(\xi_i) - \bar{\mu}_i \approx f_i'(1)(\xi_1 - 1)$. We also made use of Eqs. (15), (24), and (29). Thus, the system evolves towards the chemical equilibrium fixed point, provided the constraint (15) is satisfied.

In order to arrive at the second equality in Eq. (31), we had to take into account charge and/or particle number conservation. The equilibrium values of the chemical potentials $\bar{\mu}_i$ are given by the equilibrium value of the chemical potentials associated to conserved quantities. In the case of a single charge conservation law this is written as:

$$\bar{\mu}_i = q_i \bar{\mu} \ , \tag{32}$$

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where q_i denotes the generic charge of (anti-)particles of species "i". Thus we obtain the relation:

$$\sum_i \bar{\mu}_i \bar{\rho}_i \partial_t \xi_i = \bar{\mu} \partial_t \sum_i q_i \rho_i = 0 \ , \tag{33}$$

which we employed in Eq. (31). The generalization including several distinct abelian conservation laws is obvious, while the consideration of nonabelian charges in this context presents an open problem, cf. Ref. [24].

In passing we mention that

$$\tau^{-1} \equiv -\partial_t \ln[\mathcal{M}(\vec{\xi}, \vec{\xi}_1)] \ , \tag{34}$$

may provide a useful measure of the chemical equilibration rate. According to Eq. (31), it can be written as a sum of partial equilibration rates, $\tau^{-1} = \sum_i \tau_i^{-1}$, with $\tau_i^{-1} \propto -\mu_i \mathcal{J}_i$, while a more explicit form of the source terms, cf. Eq. (24), allows to relate it to the equilibration times of chemical rate equations.

Discussion of the First-order Formalism. Some general remarks about the first-order dissipative relativistic fluid dynamics presented in this work are in order here. It is labeled "first order", since the entropy current, Eq. (13) together with Eqs. (16)-(19), contains only up to first order gradients, which describe deviations from the equilibrium state of the fluid.

It has been argued before that in a wide class of first-order theories the equilibrium state is unstable under physically admissible perturbations [25]. This includes the theories of Eckart [26] and of Landau and Lifshitz [27] especially. Unrealistically large growth rates of the unstable modes[2] led the authors of Refs. [25] to discard these theories in favor of the Israel-Stewart second-order theory [28], where only stable or damped modes are found. The expense is the increased complexity of the theory based on a larger number of degrees of freedom. This goes together with an increased number of phenomenological parameters, which ultimately should be calculated microscopically, as we discussed.

We do not know, whether the first-order approach presented here prevents such unphysical instabilities. A full linear response analysis studying this problem is beyond the scope of this paper and is left for future work.

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[2]However, these were calculated only for macroscopic fluids, such as water. It is not obvious that the instabilities would be unrealistically fast for fluid parameters representing microscopic fluids, such as QGP droplets.

However, the growth of potentially unstable modes will be cut off by the finite size of the system in most realistic applications and by the additional dissipative processes taking place at the surface, such as radiation losses. These can be naturally incorporated in the future. Furthermore, we do take into account that the fluid may undergo dynamical changes of its composition. This is particularly important for relativistic fluids at extreme energy densities, where particle production is an important dissipative process, as long as chemical equilibrium is not fully established. Our considerations related to particle sources have shown that small perturbations of the chemical equilibrium are damped and decay. The system is driven by chemical reactions or particle production towards the equilibrium state.

Conclusions. We have presented here the theoretical considerations which allow to determine the entropy production which is consistent with the restriction to first-order space-time gradients of the thermodynamical and fluid variables, as well as with the requirements of the First and Second Laws of Thermodynamics. Our result in Eq. (1) and the subsequent hydrodynamical framework are generally applicable to special relativistic fluids. As new elements, which need to be considered in applications to high-energy nuclear collisions and QGP, we introduce here the entropy producing mechanisms of chemo- and thermo-diffusion together with the contribution of particle production.

We point out that nonlinear flow systems, such as the one describing the particle production processes in QGP, which we studied in relation to chemical equilibration, generally tend to show more structure than a simple attractive fixed point [23]. Therefore, one may speculate, whether some particular fluctuations may be caused by these nonlinearities in high-energy collisions (high multiplicity events), *i.e.* by the nonequilibrium chemistry of the reactions, with interesting consequences for the entropy production.

Summarizing, the Eqs. (1), (3), and (6), together with the relevant definitions and supplemented by an equation of state, present a consistent framework for the study of dissipative relativistic hydrodynamical systems. This comprises imperfect fluids with composition changing processes, particle production in particular, and the more familiar dissipative effects of heat conduction, shear and bulk viscosities, and diffusion. We discussed the potential shortcomings of this formalism, which involves only up to first-order space-time gradients.

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References

[1] F. Karsch, E. Laermann and A. Peikert, *Phys. Lett.* B **478** (2000) 447.

[2] C. DeTar, "Quark Gluon Plasma in Numerical Simulations of Lattice QCD", in: *Quark Gluon Plasma 2*, R.C. Hwa (Ed.) (World Sci., Singapore 1995), p.1 .

[3] J.D. Bjorken, *Phys. Rev.* D **27** (1983) 140.

[4] L.P. Csernai, *Introduction to Relativistic Heavy Ion Collisions* (Wiley, New York, 1994).

[5] For a bibliographic review of the subject and its roots see: J. Harris and B. Müller, *Ann. Rev. Nucl. Part. Sci.* **46** (1996) 71.

[6] See the web page:
http://www.cern.ch/CERN/Announcements/2000/NewStateMatter/
The text of the scientific consensus view of the spokesmen of CERN experiments is also available as E-print: nucl-th/0002042, "Evidence for a New State of Matter: An Assessment of the Results from the CERN Lead Beam Programme", compilation by U. Heinz and M. Jacob.

[7] J. Letessier *et al.*, *Phys. Rev. Lett.* **70** (1993) 3530.

[8] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972) Sects. 2-10 and 2-11.

[9] G. Baym, in: *Quark Matter '84*, edited by K. Kajantie, Lecture Notes in Physics, Vol. 221 (Springer, New York, 1985), p.39 .

[10] P. Arnold, G.D. Moore and L.G. Yaffe, *JHEP* **11** (2000) 001.

[11] J. Rafelski and B. Müller, *Phys. Rev. Lett.* **48** (1982) 1066.

[12] K. Kajantie and L. McLerran, *Nucl. Phys.* B **214** (1983) 261.

[13] K. Kajantie, R. Raitio and P.V. Ruuskanen, *Nucl. Phys.* B **222** (1983) 152.

[14] S. Hamieh, J. Letessier and J. Rafelski, *Phys. Rev.* C **62** (2000) 064901.

[15] P. Koch, B. Müller and J. Rafelski, *Phys. Rep.* **142** (1986) 167;
J. Zimányi, P. Lévai, B. Lukács and A. Rácz, in: *Particle production in Highly Excited Matter*, edited by H.H. Gutbrod and J. Rafelski, NATO ASI Series B, vol. 303 (Plenum, New York, 1993), p.243.

[16] T.S. Biró, E. van Doorn, B. Müller, M.H. Thoma and X.-N. Wang, *Phys. Rev.* C **48** (1993) 1275.

[17] J. Letessier, J. Rafelski and A. Tounsi, *Phys. Rev.* C **50** (1994) 406.

[18] L.D. Landau and E.M. Lifshitz, *Statistical Physics* (Pergamon, London, 1985) Sects. 40 and 54.

[19] M. Danos and J. Rafelski, *Phys. Lett.* B **97** (1980) 279.

[20] K. Redlich and L. Turko, *Z. Phys.* C **5** (1980) 201;
L. Turko, *Phys. Lett.* B **104** (1981) 153.

[21] H.-Th. Elze, W. Greiner and J. Rafelski, *Phys. Lett.* B **124** (1983) 515;
H.-Th. Elze and W. Greiner, *Phys. Lett.* B **179** (1986) 385.

[22] I. Mardor and B. Svetitsky, *Phys. Rev.* D **44** (1991) 878;
J. Madsen, D.M. Jensen and M.B. Christiansen, *Phys.Rev.* C **53** (1996) 1883;
D.M. Jensen and J. Madsen, *Phys. Rev.* D **53** (1996) R4719.

[23] H.G. Schuster, *Deterministic Chaos: An Introduction*, 2nd ed. (VCH, New York/Weinheim, 1989).

[24] L. Turko and J. Rafelski, *Europ. Phys. J.* C (2001), in press; [hep-th/0003079].

[25] M. Kranys, *Nuovo Cim.* **50B** (1966) 48;
D.C. Kelly, *Am. J. Phys.* **35** (1968) 585;
W.A. Hiscock and L. Lindblom, *Phys. Rev.* D **31** (1985) 725; and D **35** (1987) 3723.

[26] C. Eckart, *Phys. Rev.* **58** (1940) 919.

[27] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, Mass., 1958) Sect. 127.

[28] W. Israel, *Ann. Phys. (N.Y.)* **100** (1976) 310;

W. Israel and J.M. Stewart, *Ann. Phys. (N.Y.)* **118** (1979) 341.